Factorization (Splitting)

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FACTORIZATION

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1 INTRODUCTION

Parikh and co-authors have investigated a notion of logical independence, based on the sharing of essential propositional variables. We do a semantical analogue here. What Parikh et al. call splitting on the logical level, we call factorization (on the semantical level). Note that many of our results are valid for arbitrary products, not only for classical model sets.

We claim no originality of the basic ideas, just our proofs might be new - but they are always elementary and very easy.

The Situation:

We work here with arbitrary, non-empty products. Intuitively, $\mathcal{Y}$ is the set of models for the propositional variable set U. We assume the Axiom of Choice.

Definition 1.1

Let U be an index set, $\mathcal{Y} = \Pi\{Y_k : k \in U\}$, let all $Y_k \neq \emptyset$, and $\mathcal{X} \subseteq \mathcal{Y}$. Thus, $\sigma \in \mathcal{X}$ is a function from U to $\bigcup\{Y_k : k \in U\}$ s.t. $\sigma(k) \in Y_k$. We then note $X_k := \{y \in Y_k : \exists \sigma \in \mathcal{X}.\sigma(k) = y\}$.

If $U' \subseteq U$, then $\sigma\lceil U'$ will be the restriction of σ to U', and $\mathcal{X}\lceil U' := \{\sigma\lceil U' : \sigma \in \mathcal{X}\}$.

If $\mathcal{A} := \{A_i : i \in I\}$ is a partition of U, $U' \subseteq U$, then $\mathcal{A}\lceil U' := \{A_i \cap U' \neq \emptyset : i \in I\}$.

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Let $\mathcal{A} := \{A_i : i \in I\}$, $\mathcal{B} := \{B_j : j \in J\}$ both be partitions of U, then $\mathcal{A}$ is called a refinement of $\mathcal{B}$ iff for all $i \in I$ there is $j \in J$ s.t. $A_i \subseteq B_j$.

A partition $\mathcal{A}$ of U will be called a factorization of $\mathcal{X}$ iff $\mathcal{X} = \{\sigma \in \mathcal{Y} : \forall i \in I(\sigma\lceil A_i \in \mathcal{X}\lceil A_i)\}$, we will also sometimes say for clarity that $\mathcal{A}$ is a partition of $\mathcal{X}$ over U.

We will adhere to above notations throughout these pages.

If $\mathcal{X}$ is as above, $U' \subseteq U$, and $\sigma \in \mathcal{X}\lceil U'$, then there is obviously some (usually not unique) $\tau \in \mathcal{X}$ s.t. $\tau\lceil U' = \sigma$. This trivial fact will be used repeatedly in the following pages. We will denote by σ^+ some such τ - context will tell which are the U' and U. (To be more definite, we may take the first such τ in some arbitrary enumeration of $\mathcal{X}$.)

Given a propositional language $\mathcal{L}$, $v(\mathcal{L})$ will be the set of its propositional variables, and $v(\phi)$ the set of variables occuring in ϕ. A model set C is called definable iff there is a theory T s.t. $C = M(T)$ - the set of models of T.

2 THE RESULTS

Fact 2.1

If $\mathcal{A}$, $\mathcal{B}$ are two partitions of U, $\mathcal{A}$ a factorization of $\mathcal{X}$, and $\mathcal{A}$ a refinement of $\mathcal{B}$, then $\mathcal{B}$ is also a factorization of $\mathcal{X}$.

Proof:

Trivial by definition. $\Box$

Fact 2.2

Let $\mathcal{A}$ be a factorization of $\mathcal{X}$ over U, $U' \subseteq U$. Then $\mathcal{A}\lceil U'$ is a factorization of $\mathcal{X}\lceil U'$ over U'.

Proof:

If $A_i \cap U' \neq \emptyset$, let $\sigma'_i \in \mathcal{X}\lceil(A_i \cap U')$. Let then $\sigma_i := \sigma_i'^{+}\lceil A_i$. If $A_i \cap U' = \emptyset$, let $\sigma_i := \tau\lceil A_i$ for any $\tau \in \mathcal{X}$. Then $\sigma := \bigcup\{\sigma_i : i \in I\} \in \mathcal{X}$ by hypothesis, so $\sigma\lceil U' \in \mathcal{X}\lceil U'$, and $\sigma\lceil(A_i \cap U'\} = \sigma'_i$. $\Box$

Fact 2.3

If $A \cup A'$ is a factorization of $\mathcal{X}$ over U, $\mathcal{A}$ a factorization of $\mathcal{X}\lceil A$ over A, $\mathcal{A}'$ a factorization of $\mathcal{X}\lceil A'$ over A', then $\mathcal{A} \cup \mathcal{A}'$ is a factorization of $\mathcal{X}$ over U.

Proof:

Trivial $\square$.

Fact 2.4

If $\mathcal{A}$, $\mathcal{B}$ are two factorizations of $\mathcal{X}$, then there is a common refining factorization.

Proof:

Let σ s.t. $\forall i \in I \forall j \in J(\sigma\lceil(A_i \cap B_j) \in \mathcal{X}\lceil(A_i \cap B_j))$, show $\sigma \in \mathcal{X}$. Fix $i \in I$. By Fact 2.2, $\mathcal{B}\lceil A_i$ is a factorization of $\mathcal{X}\lceil A_i$, so $\cup\{\sigma\lceil(A_i \cap B_j) : j \in J,\ A_i \cap B_j \neq \emptyset\} = \sigma\lceil A_i \in \mathcal{X}\lceil A_i$. As $\mathcal{A}$ is a factorization of $\mathcal{X}$, $\sigma \in \mathcal{X}$. $\square$

This does not generalize to infinitely many factorizations:

Example 2.1

Take as index set $\omega + 1$, all $Y_k := \{0, 1\}$. Take $\mathcal{X} := \{\sigma : \sigma\lceil\omega$ arbitrary, and $\sigma(\omega) := 0$ iff $\sigma\lceil\omega$ is finally constant $\}$. Consider the partitions $\mathcal{A}_n := \{n, (\omega + 1) - n\}$, they are all fatorizations of $\mathcal{X}$, as it suffices to know the sequence from $n + 1$ on to know its value on ω. A common refinement $\mathcal{A}$ will have some $A \in \mathcal{A}$ s.t. $\omega \in A$. Suppose there is some $n \in \omega \cap A$, then $A \not\subseteq n + 1$, $A \not\subseteq (\omega + 1) - (n + 1)$, this is impossible, so $A = \{\omega\}$. If $\mathcal{A}$ were a factorization of $\mathcal{X}$, so would be $\{\omega, \{\omega\}\}$ by Fact 2.1, but $\mathcal{X}$ does not factor into $\mathcal{X}\lceil\omega$ and $\mathcal{X}\lceil\{\omega\}$.

Comment 2.1

Above set $\mathcal{X}$ is not definable as a model set of a corresponding language $\mathcal{L}$: If ϕ is not a tautology, there is a model m s.t. $m \models \neg\phi$. ϕ is finite, let its variables be among $p_1, \ldots, p_n$ and perhaps p_{ω}. If p_ω is not among its variables, it is trivially also false in some m' in $\mathcal{X}$. If it is, then modify m accordingly beyond n. Thus, exactly all tautologies are true in $\mathcal{X}$, but $\mathcal{X} \neq \mathcal{Y} =$ the set of all $\mathcal{L}-$models.

We have, however:

Fact 2.5

Let $\mathcal{X} = \bigcap\{\mathcal{X}_m : m \in M\}$ and $\mathcal{X}, \mathcal{X}_m \subseteq \mathcal{Y}$ for all $m \in M$.

Let $\mathcal{A}$ be a partition of U, and a factorization of all $\mathcal{X}_m$.

Then $\mathcal{A}$ is also a factorization of $\mathcal{X}$.

Proof:

Let σ s.t. $\forall i \in I$ $\sigma\lceil A_i \in \mathcal{X}\lceil A_i$.

But $\mathcal{X}\lceil A_i = (\bigcap\{\mathcal{X}_m : m \in M\})\lceil A_i \subseteq \bigcap\{\mathcal{X}_m\lceil A_i : m \in M\}$: Let $\tau \in \mathcal{X}\lceil A_i$, so by $\mathcal{X} = \bigcap\{\mathcal{X}_m : m \in M\}$ $\tau^+ \in \mathcal{X}_m$ for all $m \in M$, so $\tau \in \mathcal{X}_m\lceil A_i$ for all $m \in M$.

Thus, $\forall i \in I, \forall m \in M : \sigma\lceil A_i \in \mathcal{X}_m\lceil A_i$, so $\forall m \in M.\sigma \in \mathcal{X}_m$ by prerequisite, so $\sigma \in \mathcal{X}$. $\square$

Fact 2.6

Let $A \cup A'$ be a partition of U, and for all $\sigma \in \mathcal{X}\lceil A$ and all $\tau : A' \to \bigcup\{X_k : k \in A'\}$ with $\tau(k) \in X_k$ $\sigma \cup \tau \in \mathcal{X}$. Then

(1) $A \cup A'$ is a factorization of $\mathcal{X}$ over U.

(2) Any partition $\mathcal{A}' = \{A'_k : k \in I'\}$ of A' is a factorization of $\mathcal{X}\lceil A'$ over A'.

(3) If $\mathcal{A}$ is a factorization of $\mathcal{X}\lceil A$ over A, and $\mathcal{A}'$ a partition of A', then $\mathcal{A} \cup \mathcal{A}'$ is a factorization of $\mathcal{X}$.

Proof:

(1) and (2) are trivial, (3) follows from (1), (2), and Fact 2.3. $\square$

Corollary 2.7

Let $U = v(\mathcal{L})$ for some language $\mathcal{L}$. Let $\mathcal{X}$ be definable, and $\{\mathcal{A}_m : m \in M\}$ be a set of factorizations of $\mathcal{X}$ over U. Then $\mathcal{A} := \cup\{\mathcal{A}_m : m \in M\}$ is also a factorization of $\mathcal{X}$.

Proof:

Let $\mathcal{X} = M(T)$. Consider $\phi \in T$. $v(\phi)$ is finite, consider $\mathcal{X}\lceil v(\phi)$. There are only finitely many different ways $v(\phi)$ is partitioned by the $\mathcal{A}_m$, let them all be among $\mathcal{A}_{m_0}, \ldots, \mathcal{A}_{m_p}$. $M(\phi)\lceil v(\phi)$ might not be factorized by all $\mathcal{A}_{m_0}\lceil v(\phi), \ldots, \mathcal{A}_{m_p}\lceil v(\phi)$, but $M(T)\lceil v(\phi)$ is by Fact 2.2. By Fact 2.4, $\mathcal{A}\lceil v(\phi)$ is a factorization of $M(T)\lceil v(\phi)$.

Consider now $\mathcal{X}_\phi := (M(T) \lceil v(\phi)) \times \Pi\{(0,1) : k \in v(\mathcal{L}) - v(\phi)\}$.

By Fact 2.6, (1) $\{v(\phi), v(\mathcal{L}) - v(\phi)\}$ is a factorization of $\mathcal{X}_\phi$ over $v(\mathcal{L})$.

By Fact 2.6, (2) $\mathcal{A}\lceil(v(\mathcal{L}) - v(\phi))$ is a factorization of $\mathcal{X}_\phi\lceil(v(\mathcal{L}) - v(\phi))$ over $v(\mathcal{L}) - v(\phi)$.

By Fact 2.6, (3) $\mathcal{A}$ is a factorization of $\mathcal{X}_\phi$ over $v(\mathcal{L})$.

$M(T) = \bigcap\{(M(T)\lceil v(\phi)) \times \Pi\{(0,1) : k \in v(\mathcal{L}) - v(\phi)\}\colon\ \phi \in T\}$, so by Fact 2.5, $\mathcal{A}$ is a factorization of $M(T)$.

$\square$

Comment 2.2

Obviously, it is unimportant here that we have only 2 truth values, the proof would just as well work with any, even an infinite, number of truth values. What we really need is the fact that a formula affects only finitely many propositional variables, and the rest are free.

Remark 2.8

The Hamming distance cooperates well with factorization: Let $T \vdash \phi$, and we want to revise by $\neg\phi$. Let $M(T)$ factorize into A and A', and let ϕ not "concern" A. Then any $\phi-$model can be made to agree on A with some T-model, with an at least as good Hamming distance. (Proof: Take any $\phi-$model, modify it on A as you like, it will still be a $\phi-$model.)

Unfortunately, the manner of coding can determine if there is a factorization, as can be seen by the following example:

Example 2.2

(1) p= "*blue*", q= "*round*", q'= "*blue iff round*".

Then

$p \wedge q = blue\ and\ round,\ \neg p \wedge \neg q = \neg blue\ and\ \neg round$

$p \wedge q' = blue\ and\ round,\ \neg p \wedge q' = \neg blue\ and\ \neg round$

Thus, both code the same (meta-) situation, the first cannot be factorized, the second can.

(2)

More generally, we can code e.g. the non-factorising situation $\{p \wedge q \wedge r, \neg p \wedge \neg q \wedge \neg r\}$ also using $q' = p \leftrightarrow q$, $r' = p \leftrightarrow r$, and have then the factorising situation $\{p \wedge q \wedge r, \neg p \wedge q' \wedge r'\}$.

(3)

The following situation cannot be made factorising: $\{p \wedge q,\ p \wedge \neg q,\ \neg p \wedge \neg q\}$. Suppose there were some such solution. Then we need some p' and q', and all 4 possibilities $\{p' \wedge q',\ p' \wedge \neg q',\ \neg p' \wedge q',\ \neg p' \wedge \neg q'\}$. If we do not admit impossible situations (i.e. one of the 4 possibilities is a contradictory coding), then 2 possibilities have to contain the same situation, e.g. $p \wedge q$. But they are mutually exclusive (as they are negations), so this is impossible.

$\square$